6

Identification

The topic of this chapter corresponds to the second step of SEM: the evaluation of identification, or whether it is theoretically possible for the computer to derive a unique set of model parameter estimates. This chapter shows you how to evaluate the identification status of core types of structural equation models analyzed within single samples when means are not also estimated. A set of identification rules or heuristics is introduced. These rules describe sufficient requirements for identifying certain types of core structural equation models, and they are relatively straightforward to apply. There may be no heuristics for more complex models, but suggestions are offered about how to deal with the identification problem for such models. Some of the topics discussed next require careful and patient study. However, many examples are offered, and exercises for this chapter give you additional opportunities for practice. A Chinese proverb states that learning is a treasure that will follow you everywhere. After mastering the concepts in this chapter, you will be better prepared to apply SEM in your own studies.

GENERAL REQUIREMENTS

There are two general requirements for identifying any structural equation model. Expressed more formally, these requirements are necessary but insufficient for identification; they are:

1. The model degrees of freedom must be at least zero ($df_M \geq 0$).
2. Every latent variable (including the residual terms) must be assigned a scale (metric).

Minimum Degrees of Freedom

Some authors describe the requirement for $df_M \geq 0$ as the **counting rule** (Kaplan, 2009). Models that violate the counting rule are not identified. Specifically, they are **underidentified** or **underdetermined**. As an example of how a deficit of observations leads to nonidentification, consider the following equation:

$$a + b = 6 \tag{6.1}$$

Look at this expression as a model, the 6 as an observation, and a and b as parameters. Because Equation 6.1 has more parameters (2) than observations (1), it is impossible to find unique estimates for its parameters. In fact, there are an infinite number of solutions, including ($a = 4$, $b = 2$), ($a = 8$, $b = -2$), and so on, all of which satisfy Equation 6.1. A similar thing happens when a computer tries to derive a unique set of estimates for the parameters of an underidentified structural equation model: it is impossible to do so, and the attempt fails.

This next example shows that having equal numbers of observations and parameters does not guarantee identification. Consider the following set of formulas:

$$\begin{aligned} a + b &= 6 \\ 3a + 3b &= 18 \end{aligned} \tag{6.2}$$

Although this model has two observations (6, 18) and two parameters (a, b), it does not have a unique solution. Actually, an infinite number of solutions satisfy Equation 6.2, such as ($a = 4$, $b = 2$), ($a = 8$, $b = -2$), and so on. This happens due to an inherent characteristic of the model: the second formula in Equation 6.2 ($3a + 3b = 18$) is not unique. Instead, it is simply three times the first formula ($a + b = 6$), which means that it cannot narrow the range of solutions that satisfy the first formula. These two formulas can also be described as linearly dependent.

Now consider the following set of formulas with two observations and two parameters where the second formula is not linearly dependent on the first:

$$\begin{aligned} a + b &= 6 \\ 2a + b &= 10 \end{aligned} \tag{6.3}$$

This two-observation, two-parameter model has a unique solution ($a = 4$, $b = 2$); therefore, it is **just-identified** or **just-determined**. Note something else about Equation 6.3: given estimates of its parameters, it can perfectly reproduce the observations (6, 10). Recall that most structural equation models with zero degrees of freedom ($df_M = 0$) that are also identified can perfectly reproduce the data (sample covariances), but such models test no particular hypothesis.

A statistical model can also have fewer parameters than observations. Consider the following set of formulas with three observations and two parameters:

$$\begin{aligned} a + b &= 6 \\ 2a + b &= 10 \\ 3a + b &= 12 \end{aligned} \tag{6.4}$$

Try as you might, you will be unable to find values of a and b that satisfy all three formulas. For example, the solution ($a = 4$, $b = 2$) works only for the first two formulas in Equation 6.4, and the solution ($a = 2$, $b = 6$) works only for the last two formulas. At first, the absence of a solution seems paradoxical, but there is a way to solve this problem: Impose a statistical criterion that leads to unique estimates for an **overidentified** or **overdetermined** model with more observations than parameters. An example of such a criterion for Equation 6.4 is presented next:

> Find values of a and b that are positive and yield total scores such that the sum of the squared differences between the observations (6, 10, 12) and these totals is as small as possible.

Applying the criterion just stated to the estimation of a and b in Equation 6.4 yields a solution that not only gives the smallest squared difference (.67) but that is also unique. (Using only one decimal place, we obtain $a = 3.0$ and $b = 3.3$.) Note that this solution does not perfectly reproduce the observations (6, 10, 12) in Equation 6.4. Specifically, the three total scores obtained from Equation 6.4 given the solution ($a = 3.0$, $b = 3.3$) are (6.3, 9.3, 12.3). The fact that an overidentified model may not perfectly reproduce the data has an important role in model testing, one that is explored in later chapters.

Note that the terms *just-identified* and *overidentified* do not automatically apply to a structural equation model unless it meets both of the two necessary requirements for identification mentioned at the beginning of this section *and* additional, sufficient requirements for that particular type of model described later. That is:

1. A **just-identified structural equation model** is identified and has the same number of free parameters as observations ($df_M = 0$).
2. An **overidentified structural equation model** is identified and has fewer free parameters than observations ($df_M > 0$).

A structural equation model can be underidentified in two ways. The first case occurs when there are more free parameters than observations ($df_M < 0$). The second case happens when some model parameters are underidentified because there is not enough available information to estimate them but others are identified. In the second case, the whole model is considered nonidentified, even though its degrees of freedom could be greater than or equal to zero ($df_M \geq 0$). A general definition by Kenny (2004) that covers both cases just described is:

3. An **underidentified structural equation model** is one for which it is not possible to uniquely estimate all of its parameters.

Scaling Latent Variables

Recall that error (residual) terms in SEM can be represented in model diagrams as latent variables. Accordingly, each error term requires a scale just as every substantive latent variable (i.e., factor) must be scaled, too. Options for scaling each type of variable are considered next.

Error Terms

Scales are usually assigned to disturbances (*D*) in structural models or measurement errors (*E*) in measurement models through a **unit loading identification** (ULI) constraint. This means that the path coefficient for the direct effect of a disturbance or measurement error—the unstandardized residual path coefficient—is fixed to equal the constant 1.0. In model diagrams, this specification is represented by the numeral 1 that appears next to the direct effect of a disturbance or a measurement error on the corresponding endogenous variable. For example, the specification

$$D_{Y_1} \rightarrow Y_1 = 1.0$$

in the path analysis (PA) model of Figure 5.8(a) represents the assignment of a scale to the disturbance of endogenous variable Y_1. This specification has the consequence of assigning to D_{Y_1} a scale that is related to that of the unexplained variance of Y_1. Likewise, the specification

$$E_{X_1} \rightarrow X_1 = 1.0$$

in the CFA model of Figure 5.8(c) assigns to the error term E_{X_1} a scale related to variance in the indicator X_1 that is unexplained by the factor this indicator is supposed to reflect (*A*). Once the scale of a disturbance or measurement error is set by imposing a ULI constraint, the computer needs only to estimate its variance. If residual terms are specified as correlated (e.g., Figure 5.3(b)), then the residual covariance can be estimated, too, assuming that the model with the correlated residuals is actually identified.

The specification of *any* positive scaling constant, such as 2.1 or 17.3, would identify the variance of a residual term, but it is much more common for this constant to equal 1.0. A benefit of specifying that scaling constants are 1.0 is that for observed endogenous variables, the sum of the unstandardized residual variance and the explained variance will equal the unstandardized sample (total) variance of that endogenous variable. Also, most SEM computer programs make it easier to specify a ULI constraint for disturbances or measurement errors, or they do so by default.

Factors

Two traditional methods for scaling factors are described next. A more recent method by Little, Slegers, and Card (2006) is described later in this section. The first method is to

use the same method as for error terms, that is, by imposing ULI constraints. For a factor this means to fix the unstandardized coefficient (loading) for the direct effect on any one of its indicators to equal 1.0. Again, specification of any other positive scaling constant would do, but 1.0 is the default in most SEM computer tools. In model diagrams, this specification is represented by the numeral 1 that appears next to the direct effect of a factor on one of its indicators. The indicator with the ULI constraint is known as the **reference variable** or **marker variable**. This specification assigns to a factor a scale related to that of the explained (common, shared) variance of the reference variable. For example, the specification

$$A \rightarrow X_1 = 1.0$$

in the CFA model of Figure 6.1(a) makes X_1 the reference variable and assigns a scale to factor A based on the common variance of X_1. Assuming that scores on each multiple indicator of the same factor are equally reliable, the choice of which indicator is to be the reference variable is generally arbitrary. One reason is that the overall fit of the model to the data is usually unaffected by the selection of reference variables. Another is consistent with the domain sampling model, wherein effect (reflective) indicators of the same factor are viewed as interchangeable (Chapter 5). However, if indicator scores are not equally reliable, then it makes sense to select the indicator with the most reliable scores as the reference variable. After all factors are scaled by imposing a ULI constraint on the loading of the reference variable for each factor, the computer must then only estimate factor variances and covariances.

The second basic option to scale a factor is to fix its variance to a constant. Specification of any positive constant would do, but it is much more common to impose a **unit variance identification** (UVI) constraint. This fixes the factor variance to 1.0 and also standardizes the factor. When a factor is scaled through a UVI constraint, all factor loadings are free parameters. A UVI constraint is represented in model diagrams in this book with the numeral 1 next to the symbol for the variance of an exogenous variable (). For example, the variance of factor A is fixed to 1.0 in the CFA model of Figure 6.1(b). This specification not only assigns a scale to A, but it also implies that the loadings of all three of its indicators can be freely estimated with sample data. With the factors standardized, the computer must then only estimate the factor correlation. Note that scaling factors either through ULI or UVI constraints reduces the total number of free parameters by one for each factor.

Both methods of scaling factors in CFA (i.e., impose ULI or UVI constraints) generally result in the same overall fit of the model, but not always. A special problem known as constraint interaction occurs when the choice between either method affects overall model fit. This phenomenon is described in Chapter 9, but most of the time constraint interaction is not a problem. The choice between these two methods, then, is usually based on the relative merits of analyzing factors in standardized versus unstandardized form. When a CFA model is analyzed in a single sample, either method is probably acceptable. Fixing the variance of a factor to 1.0 to standardize it has the advantage of

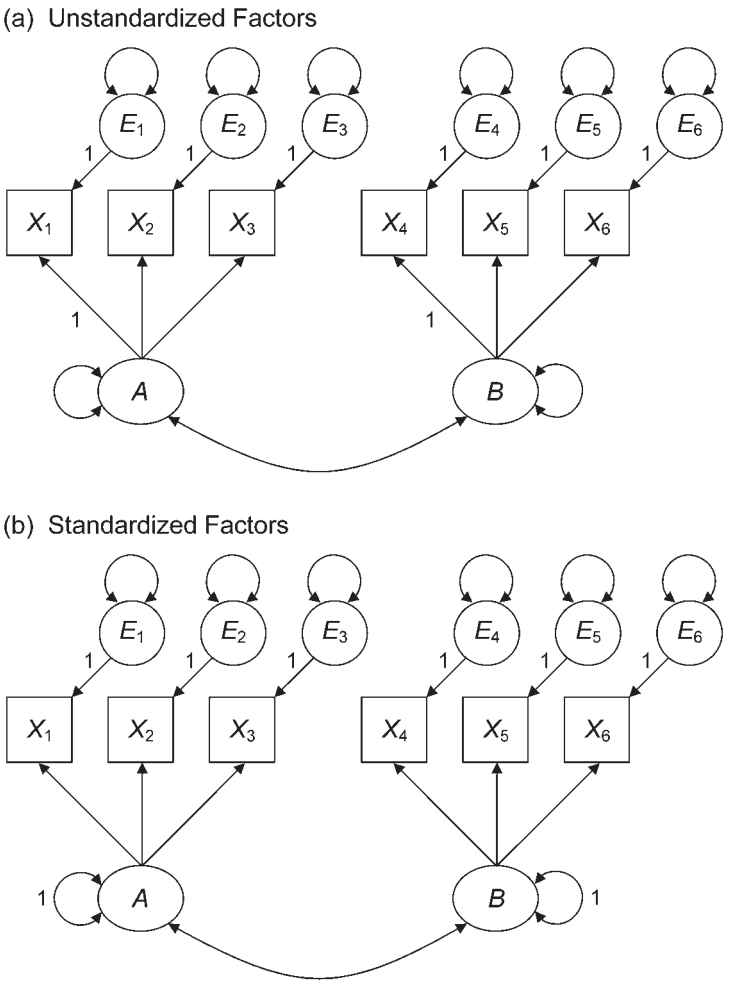

FIGURE 6.1. Standard confirmatory factor analysis measurement models with unstandardized factors (a) and standardized factors (b).

simplicity. A shortcoming of this method, however, is that it is usually applicable only to exogenous factors. This is because although basically all SEM computer tools allow the imposition of constraints on any model parameter, the variances of endogenous variables are not considered model parameters. Only some programs, such as LISREL, SEPATH, and RAMONA, allow the *predicted* variances of endogenous factors to be constrained to 1.0. This is not an issue for CFA models, wherein all factors are exogenous, but it can be for structural regression (SR) models, wherein some factors are endogenous.

There are times when standardizing factors is *not* appropriate. These include (1) the analysis of a structural equation model across independent samples that differ in their variabilities and (2) longitudinal measurement of variables that show increasing (or decreasing) variabilities over time. In both cases, important information may be lost

when factors are standardized. How to appropriately scale factors in a multiple-sample CFA analysis is considered in Chapter 9.

Exogenous factors in SR models can be scaled by imposing either a ULI constraint where the loading of one indicator per factor is fixed to 1.0 (the factor is unstandardized) or a UVI constraint where the factor variance is fixed to 1.0 (the factor is standardized). As mentioned, though, most SEM computer programs allow only the first method just mentioned for scaling endogenous factors. This implies that endogenous factors are unstandardized in most analyses. When an SR model is analyzed within a single sample, the choice between scaling an exogenous factor with either ULI or UVI constraints combined with the use of ULI constraints only to scale endogenous factors usually makes no difference. An exception is when some factors have only two indicators and there is constraint interaction, which for SR models is considered in Chapter 10.

Little, Slegers, and Card (2006) describe a third method for scaling factors in models where (1) all indicators of each factor have the same scale (i.e., range of scores) and (2) most indicators are specified to measure (load on) a single factor. This method does *not* require the selection of a reference variable, such as when ULI constraints are imposed, nor does it standardize factors, such as when UVI constraints are imposed. Instead, this third method for scaling factors relies on the capability of modern SEM computer tools to impose constraints on a set of two or more model parameters, in this case the unstandardized factor loadings of all the indicators for the same factor. Specifically, the researcher scales factors in the Little–Sleger–Card (LSC) method by instructing the computer to constrain the average (mean) loading of a set of indicators on their common factor to equal 1.0 in the unstandardized solution. So scaled, the variance of the factor will be estimated as the average explained variance across all the indicators in their original metric, weighted by the degree to which each indicator contributes to factor measurement. Thus, factors are not standardized in this method, nor does the explained variance of any arbitrarily selected indicator (i.e., that of the reference variance when imposing a ULI constraint) determine factor variance. The LSC method results in the same overall fit of the entire model to the data as observed when imposing either ULI or UVI constraints to scale factors. Also, the LSC method is appropriate for the analysis of a model in a single group, across multiple groups, or across multiple occasions (i.e., repeated measures)—see Little, Slegers, and Card (2006) for more information.

UNIQUE ESTIMATES

This is the penultimate aspect of identification: It must be possible to express each and every model parameter as a unique function of elements of the population covariance matrix such that the statistical criterion to be minimized in the analysis is also satisfied. Because we typically estimate the population covariance matrix with the sample covariance matrix, this facet of identification can be described by saying that there is a unique set of parameter estimates, given the data and the statistical criterion to be minimized.

Determining whether the parameters can be expressed as unique functions of the

sample data is *not* an empirical question. Instead, it is a mathematical or theoretical question that can be evaluated by resolving equations that represent the parameters in terms of symbols that correspond to elements of the sample covariance matrix. This exercise takes the form of a formal mathematical proof, so no actual numerical values are needed for elements of the sample covariance matrix, just symbolic representations of them. *This means that model identification can—and should—be evaluated before the data are collected.* You may have seen formal mathematical proofs for ordinary least squares (OLS) estimation in multiple regression (MR). These proofs involve showing that standard formulas for regression coefficients and intercepts (e.g., Equations 2.5, 2.7, 2.8) are, in fact, those that satisfy the least squares criterion. A typical proof involves working with second derivatives for the function to be minimized. Dunn (2005) describes a less conventional proof for OLS estimation based on the Cauchy–Schwartz inequality, which is related to the triangle inequality in geometry as well as to limits on the bounds of correlation and covariance statistics in positive-definite data matrices (Chapter 3).

The derivation of a formal proof for a simple regression analysis would be a fairly daunting task for those without a strong mathematics background, and models analyzed in SEM are often more complicated than simple regression models. Also, the default estimation method in SEM, maximum likelihood (ML), is more complex than OLS estimation, which implies that the statistical criterion minimized in ML estimation is more complicated, too. Unfortunately, SEM computer tools are of little help in determining whether or not a particular structural equation model is identified. Some of these programs perform rudimentary checks for identification, such as applying the counting rule, but these checks generally concern necessary conditions, not sufficient ones.

It may surprise you to learn that SEM computer tools are rather helpless in this regard, but there is a simple explanation: Computers are very good at *numerical* processing. However, it is harder to get them to process symbols, and it is *symbolic* processing that is needed for determining whether a particular model is identified. Computer languages for symbolic processing, such as LISP (list processing), form the basis of some applications of computers in the areas of artificial intelligence and expert systems. But contemporary SEM computer tools lack any real capability for symbolic processing of the kind needed to prove model identification for a wide range of models.

Fortunately, one does not need to be a mathematician in order to deal with the identification problem in SEM. This is because a series of less formal rules, or **identification heuristics**, can be applied by ordinary mortals (the rest of us) to determine whether certain types of models are identified. These heuristics cover many, but not all, kinds of core structural equation models considered in this part of the book. They are described next for PA models, CFA models, and fully latent SR models. This discussion assumes that the two necessary requirements for identification ($df_M \geq 0$; latent variables scaled) are satisfied. Recall that CFA models assume reflective measurement where indicators are specified as caused by the factors (Chapter 5). Formative measurement models in which underlying observed or latent composites are specified as caused by their indicators have special identification requirements that are considered in Chapter 10.

It is frustrating that computers are of little help in dealing with identification in SEM, but you can apply heuristics to verify the identification status of many types of models. Copyright 2004 by Betsy Streeter. Reprinted with permission from CartoonStock Ltd. (*www.cartoonstock.com*).

RULE FOR RECURSIVE STRUCTURAL MODELS

Because of their particular characteristics, recursive path models are always identified (e.g., Bollen, 1989, pp. 95–98). This property is even more general: Recursive structural models are identified, whether the structural model consists of observed variables only (path models) or factors only (the structural part of a fully latent SR model). Note that whether the measurement component of an SR model with a recursive structural model is also identified is a separate question, one that is dealt with later in this chapter. The facts just reviewed underlie the following sufficient condition for identification:

Recursive structural models are identified.	(Rule 6.1)

RULES FOR NONRECURSIVE STRUCTURAL MODELS

The material covered in this section is more difficult, and so readers interested in recursive structural models only can skip it (i.e., go the section on CFA). However, you can specify and test an even wider range of hypotheses about direct and indirect effects (e.g.,

feedback loops) if you know something about nonrecursive structural models, so the effort is worthwhile.

The case concerning identification for nonrecursive structural models—whether among observed variables (path models) or factors (SR models)—is more complicated. This is because, unlike recursive models, nonrecursive models are not always identified. Although algebraic means can be used to determine whether the parameters of a nonrecursive model can be expressed as unique functions of its observations (e.g., Berry, 1984, pp. 27–35), these techniques are practical only for very simple models. Fortunately, there are alternatives that involve determining whether a nonrecursive model meets certain requirements for identification that can be checked by hand (i.e., heuristics). Some of these requirements are only necessary for identification, which means that satisfying them does not guarantee identification. If a nonrecursive model satisfies a sufficient condition, however, then it is identified. These requirements are described next for nonrecursive path models, but the same principles apply to SR models with nonrecursive structural components.

The nature and number of conditions for identification that a nonrecursive model must satisfy depend on its pattern of disturbance correlations. Specifically, the necessary order condition and the sufficient rank condition apply to models with unanalyzed associations between all pairs of disturbances either for the whole model or within blocks of endogenous variables that are recursively related to each other. Consider the two nonrecursive path models in Figure 6.2. For both models, $df_M \geq 0$ and all latent variables are scaled, but these facts are not sufficient to identify either model. The model of Figure 6.2(a) has an indirect feedback loop that involves Y_1–Y_3 and all possible disturbance correlations (3). The model of Figure 6.2(b) has two direct feedback loops and a pattern of disturbance correlations described by some authors as **block recursive**. One can partition the endogenous variables of this model into two blocks, one with Y_1 and Y_2 and the other made up of Y_3 and Y_4. Each block contains all possible disturbance correlations ($D_1 \curvearrowright D_2$ for the first block, $D_3 \curvearrowright D_4$ for the second), but the disturbances across the blocks are independent (e.g., D_1 is uncorrelated with D_3). Also, the pattern of direct effects within each block is nonrecursive (e.g., $Y_1 \rightleftarrows Y_2$), but effects between the blocks are unidirectional (recursive). Thus, the two blocks of endogenous variables in the model of Figure 6.2(b) are recursively related to each other even though the whole model is nonrecursive.

Order Condition

The **order condition** is a counting rule applied to each endogenous variable in a nonrecursive model that either has all possible disturbance correlations or that is block recursive. If the order condition is not satisfied, the equation for that endogenous variable is underidentified. One evaluates the order condition by tallying the number of variables in the structural model (except disturbances) that have direct effects on each endogenous variable versus the number that do not; let's call the latter *excluded variables*. The order condition can be stated as follows:

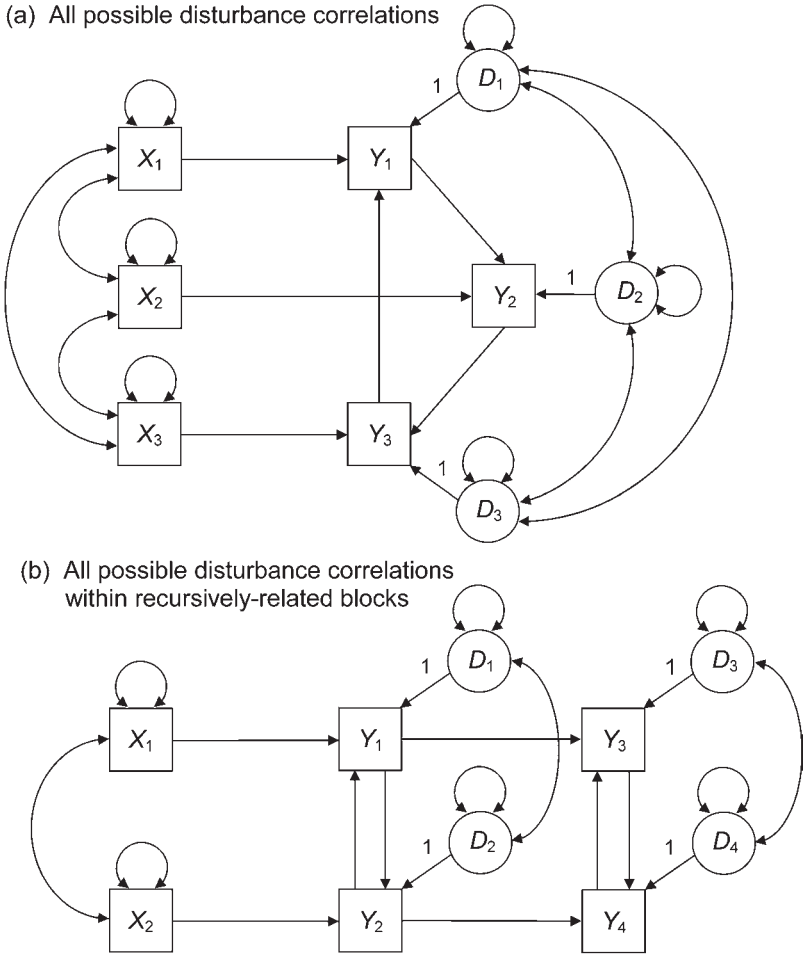

FIGURE 6.2. Two examples of nonrecursive path models with feedback loops.

> The order condition requires that the number of excluded variables for each endogenous variable equals or exceeds the total number of endogenous variables minus 1. (Rule 6.2)

For nonrecursive models with correlations between all pairs of disturbances, the total number of endogenous variables equals that for the whole model. For example, the model of Figure 6.2(a) has all possible disturbance correlations, so the total number of endogenous variables equals 3. This means that a minimum of 3 – 1 = 2 variables must be excluded from the equation of each endogenous variable, which is true here: There are three variables excluded from the equation of every endogenous variable (e.g., X_2, X_3, and Y_2 for Y_1), which exceeds the minimum number (2). Thus, the model of Figure 6.2(a) meets the order condition.

For nonrecursive models that are block recursive, however, the total number of

endogenous variables is counted separately for each block when the order condition is evaluated. For example, there are two recursively related blocks of endogenous variables in the model of Figure 6.2(b). Each block has two variables, so the total number of endogenous variables for each block is 2. To satisfy the order condition, at least 2 – 1 = 1 variables must be excluded from the equation of each endogenous variable in both blocks, which is true here. Specifically, one variable is excluded from each equation for Y_1 and Y_2 in the first block (e.g., X_2 for Y_1), and three variables are excluded from each equation for Y_3 and Y_4 in the second block (e.g., X_1, X_2, and Y_2 for Y_3). Because the number of excluded variables for each endogenous variable in every block exceeds the minimum number, the order condition is satisfied for this model.

Rank Condition

Because the order condition is only necessary, we still do not know whether the nonrecursive models in Figure 6.2 are identified. Evaluation of the sufficient **rank condition**, however, will provide the answer. The rank condition is usually described in the SEM literature in matrix terms (e.g., Bollen, 1989, pp. 98–103), which is fine for those familiar with linear algebra but otherwise not. Berry (1984) devised an algorithm for checking the rank condition that does not require extensive knowledge of matrix operations, a simpler version of which is described in Appendix 6.A. A nontechnical description of the rank condition is given next.

For nonrecursive models with all possible disturbance correlations, the rank condition can be viewed as a requirement that each variable in a feedback loop has a unique pattern of direct effects on it from variables outside the loop. Such a pattern of direct effects provides a "statistical anchor" so that the parameters of variables involved in feedback loops can be estimated distinctly from one another. Look again at Figure 6.2(a). Each of the three endogenous variables of this model has a unique pattern of direct effects on it from variables external to their indirect feedback loop; that is:

$$X_1 \rightarrow Y_1, \quad X_2 \rightarrow Y_2, \quad \text{and } X_3 \rightarrow Y_3$$

This analogy does not hold for those models considered in this book to be nonrecursive that do not have feedback loops, such as partially recursive models with correlated disturbances in a bow pattern (e.g., Figure 5.3(d)). Therefore, a more formal means of evaluating the rank condition is needed; see Appendix 6.A. The identification rule for the rank condition for nonrecursive models that either have all possible disturbance correlations or that are block recursive is stated next:

Nonrecursive models that satisfy the rank condition are identified. (Rule 6.3)

Rigdon (1995) describes a graphical technique for evaluating identification status that breaks the model down into a series of two-equation nonrecursive blocks, such as for a direct feedback loop. This graphical technique could complement or in some

cases replace evaluation of the order condition and the rank condition using the methods described here. Eusebi (2008) describes a graphical counterpart of the rank condition, but it requires knowledge of undirected, directed, and directed acyclic graphs from graphical models theory.

Respecification of Nonidentified Nonrecursive Models

Now let's consider a nonrecursive model that is not identified and some options for its respecification. Presented in Figure 6.3 is a nonrecursive path model with all possible disturbance correlations based on an example by Berry (1984). In this model, let Y_1 and Y_2 represent, respectively, violence on the part of protesters and police. The direct feedback loop in this model reflects the hypothesis that as protesters become more violent, so do the police, and vice versa. The two measured exogenous variables, X_1 and X_2, represent, respectively, the seriousness of the civil disobedience committed by the protesters and the availability of police riot gear (clubs, tear gas, etc.). Immediately after its specification but before the data are collected, the researcher evaluates its identification status. Two problems are discovered: the model has more parameters (11) than observations (10), and the order condition is violated because there are no excluded variables for Y_2. Because this model fails the order condition, it will also fail the rank condition. An exercise will ask you to verify that $df_M = -1$ for the model of Figure 6.3 and also that it fails both the order condition and the rank condition.

What can be done about this identification problem? Because the data are not yet collected, one possibility is to add exogenous variables to the model such that (1) the number of additional observations afforded by adding variables is greater than the number of free parameters they bring to the model; (2) the number of excluded variables for Y_1 and Y_2 are each at least 1; and (3) the respecified model also meets the rank condition. Suppose that it is decided that a new exogenous variable, X_3, would be protesters' level of

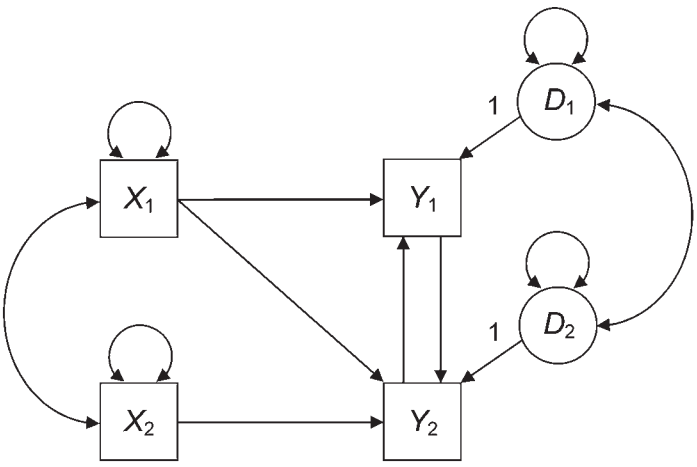

FIGURE 6.3. A nonrecursive model that is not identified.

commitment to nonviolence. The addition of the path $X_3 \rightarrow Y_1$ (Y_1 is protester violence) and unanalyzed associations between X_3 and the other two exogenous variables would accomplish the goals just listed. Thus, the model respecified in this way is identified. An exercise will ask you to verify this fact.

Equality and Proportionality Constraints

The imposition of an equality or a proportionality constraint on the direct effects of a feedback loop is one way to reduce the number of free parameters without dropping paths. For example, the specification that both direct effects of the reciprocal relation $Y_1 \rightleftarrows Y_2$ are equal means that only one path coefficient is needed rather than two. A possible drawback of imposing equality constraints on feedback loops is that they preclude the detection of unequal mutual influence. For example, Wagner, Torgeson, and Rashotte (1994) found in longitudinal studies that the effect of children's phonological processing abilities on their reading skills is about three times the magnitude of the effect in the opposite direction. If equality constraints were blindly imposed when bidirectional effects differ in magnitude, then not only may the model poorly fit the data but the researcher may miss an important finding. In contrast, a proportionality constraint allows for unequal mutual influence but on an a priori basis. For instance, it may be specified that the path $Y_1 \rightarrow Y_2$ must be three times the value of that for the path $Y_2 \rightarrow Y_1$. Like equality constraints, proportionality constraints reduce the number of free parameters, one for each pair of direct effects. However, the imposition of proportionality constraints generally requires knowledge about relative effect magnitudes.

"None-of-the-Above" Nonrecursive Models

If a nonrecursive structural model has either no disturbance correlations or less than all possible disturbance correlations such that the model is not block recursive, the order and rank conditions are generally too conservative. That is, such "none-of-the-above" nonrecursive models that fail either condition may nevertheless be identified. Unfortunately, there may be no sufficient condition that can be readily evaluated by hand to determine whether a none-of-the-above nonrecursive model is actually identified. Thus, the identification status of such models may be ambiguous. How to deal with structural equation models where identification status is unknown is discussed later.

RULES FOR STANDARD CFA MODELS

Meeting both necessary requirements also does not guarantee that a CFA measurement model is identified. For standard CFA models that specify unidimensional measurement—every indicator loads on just one factor and there are no measurement error correlations—there are some straightforward rules that concern minimum numbers of indicators per factor. They are summarized next:

If a standard CFA model with a single factor has at least three indicators, the model is identified.	(Rule 6.4)

If a standard CFA model with ≥ 2 factors has ≥ 2 indicators per factor, the model is identified.	(Rule 6.5)

That's it. The first heuristic just listed for single-factor models is known as the **three-indicator rule**, and the second heuristic for models with multiple constructs is the **two-indicator rule**. Recall that CFA models (and SR models, too) with factors that have only two indicators are more prone to problems in the analysis. It is better to have at least three to four indicators per factor to prevent such problems, but two indicators per factor is the minimum for identification.

Let's apply the requirements just discussed to the standard CFA models presented in Figure 6.4. The model of Figure 6.4(a) has a single factor with two indicators. This model is underidentified: With two observed variables, there are three observations but four parameters, including three variances of exogenous variables (of factor *A* and two measurement errors, E_1 and E_2) and one factor loading (of X_2; the other is fixed to 1.0 to scale *A*), so $df_M = -1$ for the model in Figure 6.4(a). The imposition of a constraint, such as one of equality, or

$$A \rightarrow X_1 = A \rightarrow X_2 = 1.0$$

may make this model estimable because df_M would be zero in the respecified one-factor, two-indicator model. For such models Kenny (1979) noted that if the correlation between the two indicators is negative, then the just-identified model that results by imposing an equality constraint on the factor loadings does not exactly reproduce the correlation. This is an example of a just-identified structural equation model that does not perfectly fit the data.

Because the single-factor model in Figure 6.4(b) has three indicators, it is identified. Specifically, it is just-identified: There are 3(4)/2 = 6 observations available to estimate the six-model parameters, including four variances (of factor *A* and three measurement errors) and two factor loadings ($df_M = 0$). Note that a standard, one-factor CFA model must have at least four indicators in order to be overidentified. Because each of the two factors in the model of Figure 6.4(c) has two indicators, it is identified. Specifically, it is overidentified and $df_M = 1$.

RULES FOR NONSTANDARD CFA MODELS

There is a different—and more complicated—set of rules for nonstandard CFA models that specify multidimensional measurement where some indicators load on more than a single factor or some error terms covary. Readers interested in standard CFA models

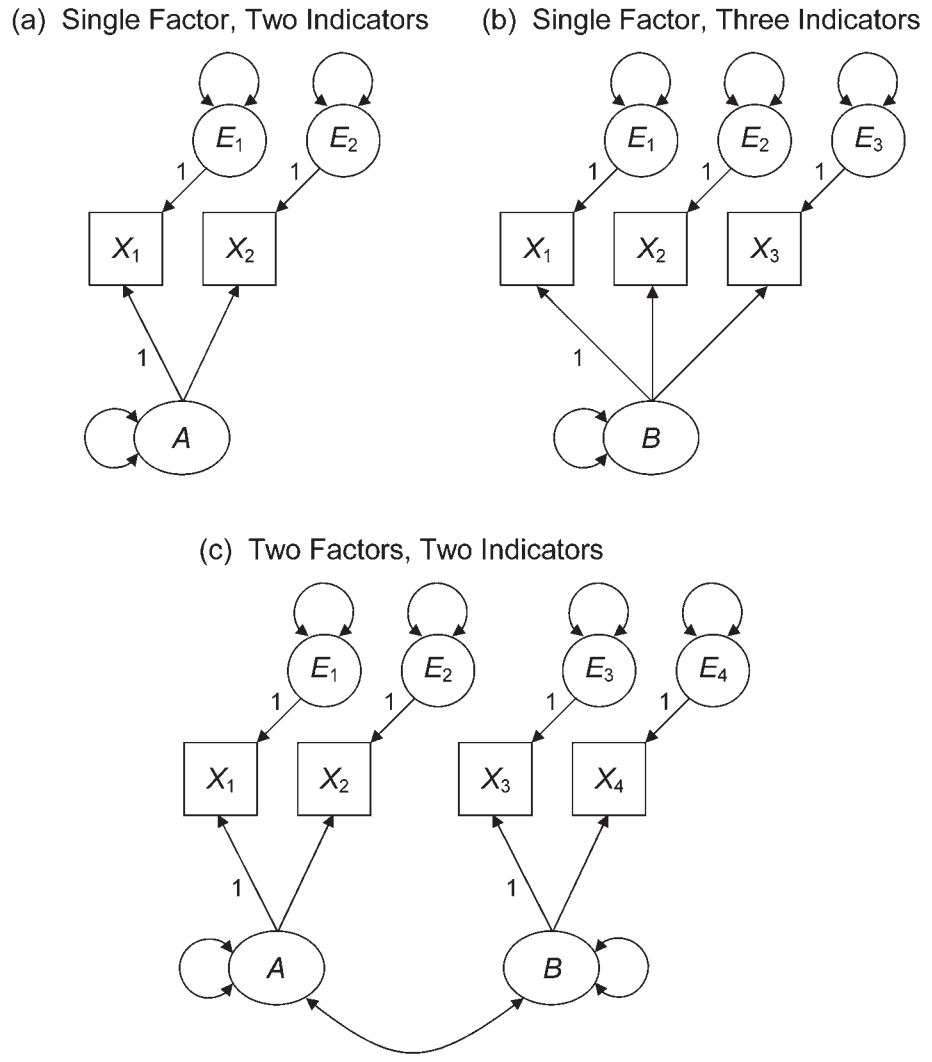

FIGURE 6.4. Identification status of three standard confirmatory factor analysis models.

only can skip this section (i.e., go to the section on SR models), but standard CFA models have more restrictive assumptions compared with nonstandard CFA models. Again, the reward of greater flexibility in hypothesis testing requires even more careful study, but you can do it.

O'Brien (1994) describes a set of rules for nonstandard measurement models where every indicator loads on a single factor but some measurement error correlations are freely estimated. These rules are applied "backwards" starting from patterns of independent (uncorrelated) pairs of error terms to prove the identification of factor loadings, then of error variances, next of factor correlations in multiple-factor models, and finally of measurement error correlations. The O'Brien rules work well for relatively simple

measurement models, but they can be awkward to apply to more complex models. A different set of identification rules by Kenny, Kashy, and Bolger (1998) that may be easier to apply is listed in Table 6.1 as Rule 6.6. This rule spells out requirements that must be satisfied by each factor (Rule 6.6a), pair of factors (Rule 6.6b), and indicator (Rule 6.6c) in order to identify measurement models with error correlations.

Rule 6.6a in Table 6.1 is a requirement for a minimum number of indicators per factor, either two or three depending on the pattern of error correlations or constraints imposed on factor loadings. Rule 6.6b refers to the specification that for every pair of factors, there must be at least two indicators, one from each factor, whose error terms are not correlated. Rule 6.6c concerns the requirement for every indicator that there is at least one other indicator in the model with which it does not share an error correlation. Rule 6.6 in Table 6.1 assumes that all factor covariances are free parameters and that there are multiple indicators of every factor. Kenny et al. (1998) describe additional rules not considered here for exceptions to these assumptions.

Kenny et al. (1998) also describe identification rules for indicators in nonstandard measurement models that load on ≥ 2 factors. Let's refer to such indicators as **complex indicators**. The first requirement is listed in the top part of Table 6.2 as Rule 6.7, and it concerns sufficient requirements for identification of the multiple-factor loadings of a complex indicator. Basically, this rule requires that each factor on which a complex indicator loads has a sufficient number of indicators (i.e., each factor meets Rule 6.6a in Table 6.1). Rule 6.7 also requires that each one of every pair of such factors has an indicator that does not share an error correlation with a corresponding indicator of the other factor (see Table 6.2). If a complex indicator shares error correlations with other indicators, then the additional requirement listed as Rule 6.8 in Table 6.2 must also be

TABLE 6.1. Identification Rule 6.6 for Nonstandard Confirmatory Factor Analysis Models with Measurement Errors

For a nonstandard CFA model with measurement error correlations to be identified, all three of the conditions listed next must hold:	(Rule 6.6)
For each factor, at least one of the following must hold:	(Rule 6.6a)
1. There are at least three indicators whose errors are uncorrelated with each other.	
2. There are at least two indicators whose errors are uncorrelated and either	
a. the errors of both indicators are not correlated with the error term of a third indicator for a different factor, or	
b. an equality constraint is imposed on the loadings of the two indicators.	
For every pair of factors, there are at least two indicators, one from each factor, whose error terms are uncorrelated.	(Rule 6.6b)
For every indicator, there is at least one other indicator (not necessarily of the same factor) with which its error term is not correlated.	(Rule 6.6c)

Note. These requirements are described as Conditions B–D in Kenny, Kashy, and Bolger (1998, pp. 253–254).

TABLE 6.2. Identification Rule 6.7 for Multiple Loadings of Complex Indicators in Nonstandard Confirmatory Factor Analysis Models and Rule 6.8 for Error Correlations of Complex Indicators

Factor loadings	
For every complex indicator in a nonstandard CFA model:	(Rule 6.7)
In order for the *multiple factor loadings* to be identified, both of the following must hold:	
1. *Each factor on which the complex indicator loads* must satisfy Rule 6.6a for a minimum number of indicators.	
2. *Every pair of those factors* must satisfy Rule 6.6b that each factor has an indicator that does not have an error correlation with a corresponding indicator on the other factor of that pair.	
Error correlations	
In order for *error correlations* that involve complex indicators to be identified, both of the following must hold:	(Rule 6.8)
1. Rule 6.7 is satisfied.	
2. For each factor on which a complex indicator loads, there must be at least one indicator with a single loading that does not have an error correlation with the complex indicator.	

Note. These requirements are described as Condition E in Kenny, Kashy, and Bolger (1998, p. 254).

satisfied, too. This rule requires that for each factor on which a complex indicator loads, there is at least one other indicator with a single loading that does not share an error correlation with the complex indicator. The requirements of Rules 6.6 and 6.7 are typically addressed by specifying that some indicators load on just a single factor.

Let's apply the identification heuristics just discussed to the nonstandard CFA models presented in Figure 6.5. To save space, I use a compact notation in the figure where latent constructs are denoted by circles, indicators by *X*s, and error terms by *E*s. However, do not forget the variance parameter associated with each exogenous variable in Figure 6.5 that is normally represented by the ↶ symbol in model diagrams elsewhere in this book. The single-factor, four-indicator model in Figure 6.5(a) has two error correlations, or

$$E_{X_2} \curvearrowright E_{X_4} \quad \text{and} \quad E_{X_3} \curvearrowright E_{X_4}$$

This model is just-identified because it has no degrees of freedom ($df_M = 0$), its factor (*A*) has at least three indicators (X_1–X_3) whose error terms are uncorrelated (Rule 6.6a), and all other requirements of Rule 6.6 (Table 6.1) are met. The single-factor, four-indicator model in Figure 6.5(b) also has two error correlations (i.e., $df_M = 0$) but in a different pattern, or

$$E_{X_1} \curvearrowright E_{X_2} \quad \text{and} \quad E_{X_3} \curvearrowright E_{X_4}$$

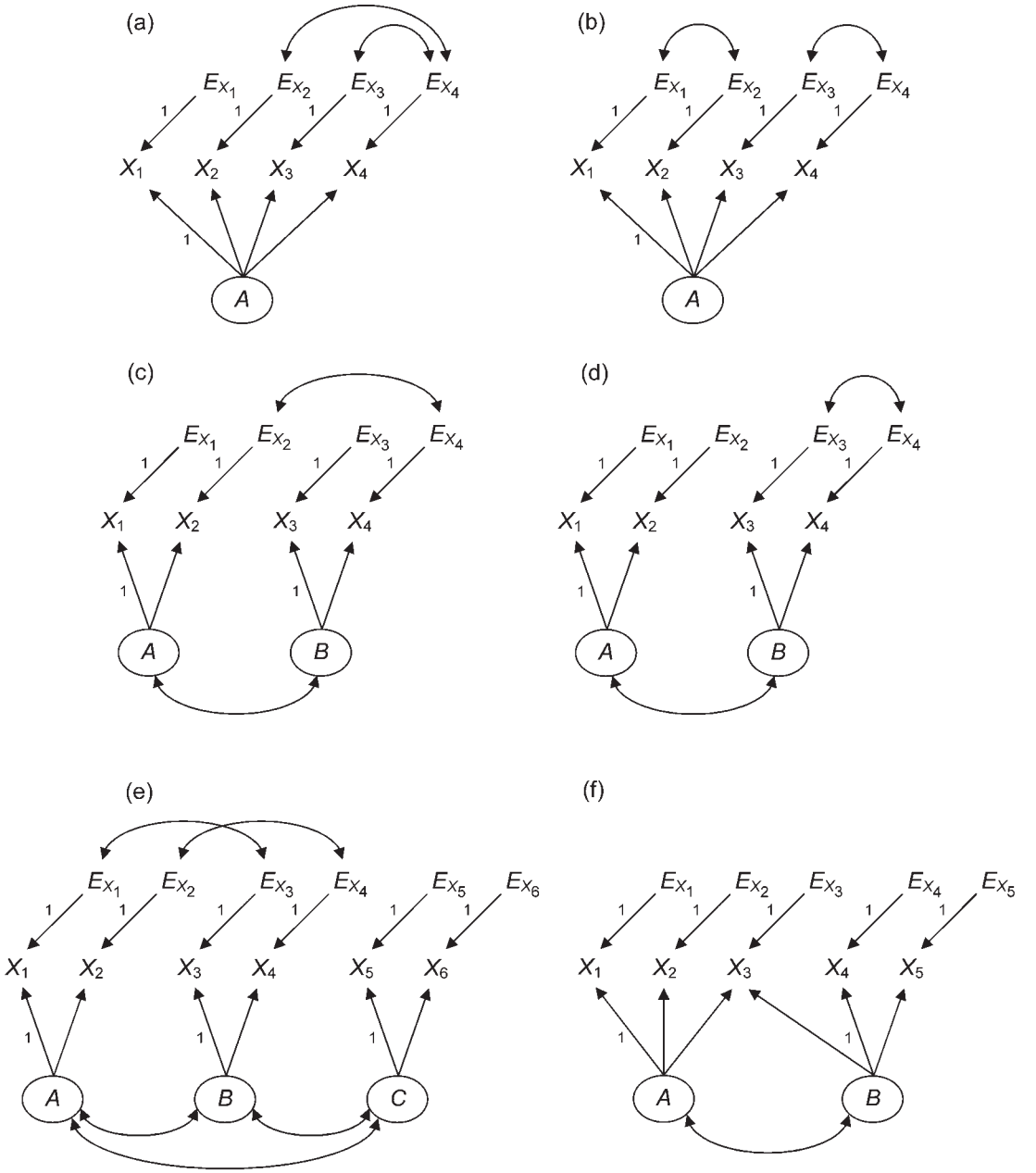

FIGURE 6.5. Identification status of nonstandard confirmatory factor analysis models.

Although this model has at least two indicators whose error terms are independent, such as X_2 and X_3, it nevertheless fails Rule 6.6a because there is no indicator of a different factor with which X_2 and X_3 do not share an error correlation. Therefore, the model in Figure 6.5(b) is not identified. However, this model would be identified if an equality constraint were imposed on the factor loadings of X_2 and X_3. That is, the specification that

$$A \rightarrow X_2 = A \rightarrow X_3$$

would be sufficient to identify the model in Figure 6.5(b) because then Rule 6.6 would be met.

The two-factor, four-indicator model of Figure 6.5(c) with a single error correlation ($E_{X_2} \curvearrowright E_{X_4}$) is just-identified because $df_M = 0$ and all three requirements for Rule 6.6 are satisfied (Table 6.1). However, the two-factor, four-indicator model in Figure 6.5(d) with a different error correlation ($E_{X_3} \curvearrowright E_{X_4}$) is not identified because it violates Rule 6.6a. Specifically, factor *B* in this model does not have two indicators whose error terms are independent. In general, it is easier to uniquely estimate cross-factor error correlations (e.g., Figure 6.5(c)) than within-factor error correlations (e.g., Figure 6.5(d)) when there are only two indicators per factor without imposing additional constraints. The three-factor, two-indicator model in Figure 6.5(e) with two cross-factor error correlations, or

$$E_{X_1} \curvearrowright E_{X_3} \quad \text{and} \quad E_{X_2} \curvearrowright E_{X_4}$$

is overidentified because the degrees of freedom are positive ($df_M = 4$) and Rule 6.6 is satisfied. This model also demonstrates that adding indicators—along with a third factor—allows the estimation of additional error correlations compared with the two-factor model in Figure 6.5(c). The model in Figure 6.5(f) has a complex indicator that loads on two factors, or

$$A \rightarrow X_3 \quad \text{and} \quad B \rightarrow X_3$$

Because this model meets the requirements of Rule 6.7 and has positive degrees of freedom ($df_M = 3$), it is overidentified. An exercise will ask you to add error correlations to this model with a complex indicator and then evaluate Rule 6.8 in order to determine whether the respecified models is identified.

The specification of either correlated measurement errors or of some indicators loading on multiple factors may not cause identification problems. The presence of both in the same model, though, can complicate matters. For example, it can be difficult to correctly apply the O'Brien rules or Kenny–Kashy–Bolger rules to complex models, especially models where some factors have at least five indicators. Because these requirements are sufficient, a complex nonstandard CFA model that is really identified could nevertheless fail some of these rules. Fortunately, most CFA models described in the

literature do not have complex indicators, so only Rule 6.6 for error correlations in measurement models is applied most often in practice.

RULES FOR SR MODELS

This section deals with fully latent SR models in which each variable in the structural model (except disturbances) is a factor measured by multiple indicators. The identification status of partially latent SR models where at least one construct in the structural model is measured by a single indicator is considered in Chapter 10. If one understands something about the identification of structural models and measurement models, there is relatively little new to learn about SR models. This is because the evaluation of whether an SR model is identified is conducted separately for each part of the model, measurement and structural. Indeed, a theme of this evaluation is that a valid (i.e., identified) measurement model is needed before it makes sense to evaluate the structural part of an SR model.

As with CFA models, meeting the two necessary requirements does not guarantee the identification of an SR model. Additional requirements reflect the view that the analysis of an SR model is essentially a path analysis conducted with estimated variances and covariances among the factors. Thus, it must be possible for the computer to derive unique estimates of the factor variances and covariances before specific direct effects among them can be estimated. In order for the structural portion of an SR model to be identified then, its measurement portion must be identified. Bollen (1989) describes this requirement as the **two-step rule**, and the steps to evaluate it are outlined next:

In order for an SR model to be identified, both of the following must hold: (Rule 6.9)

1. The measurement part of the model respecified as a CFA model is identified (evaluate the measurement model against Rules 6.4–6.8).
2. The structural part of the model is identified (evaluate the structural model against Rules 6.1–6.3).

The two-step rule is a sufficient condition: SR models that satisfy both parts of this rule are identified. Evaluation of the two-step rule is demonstrated next for the fully latent SR model presented in Figure 6.6(a). This model meets the necessary requirements because every latent variable is scaled and there are more observations than free parameters. Specifically, with six observed variables, there are 6(7)/2 = 21 observations available to estimate this model's 14 parameters, including nine variances of exogenous variables (of six measurement errors, one exogenous factor *A*, and two disturbances), three factor loadings, and two direct effects between factors ($df_M = 7$). However, we still do not know whether the model of Figure 6.6(a) is identified. To find out, we can apply the two-step

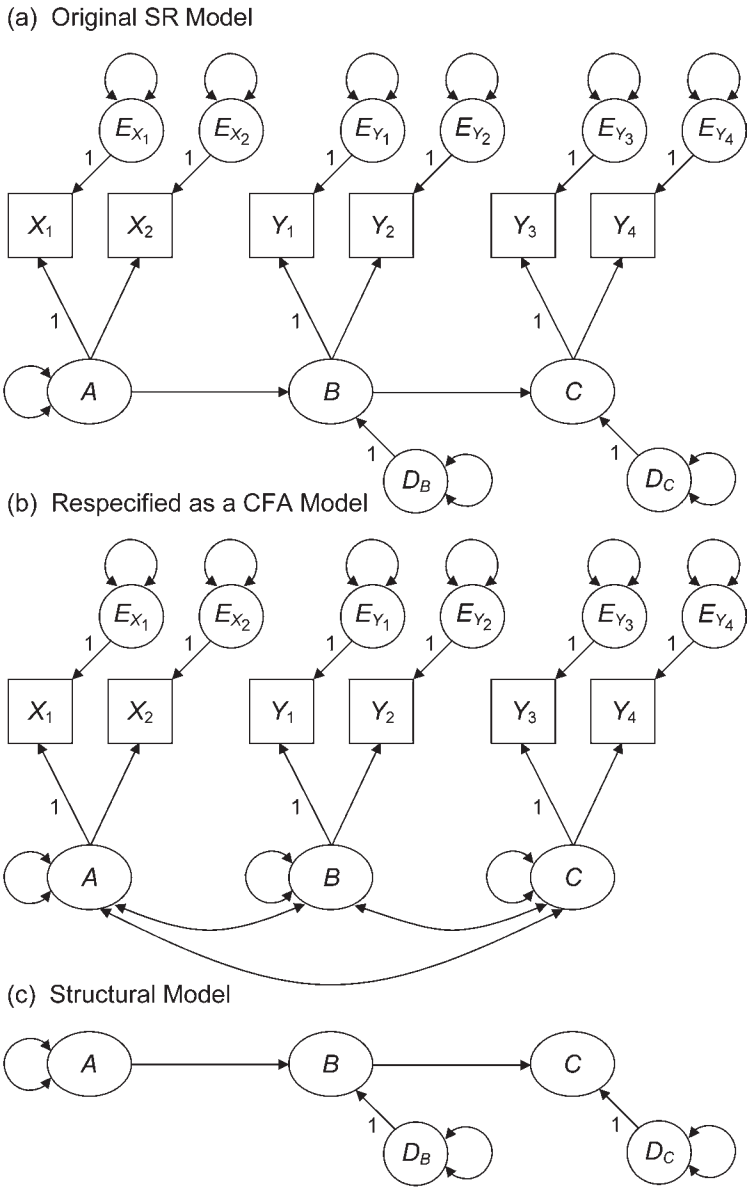

FIGURE 6.6. Evaluation of the two-step rule for identification for a fully latent structural regression (SR) model.

rule. The respecification of this SR model as a CFA measurement model is presented in Figure 6.6(b). Because this standard three-factor CFA model has at least two indicators per factor, it is identified (Rule 6.5). The first part of the two-step rule is satisfied. The structural part of the SR model is presented in Figure 6.6(c). Because the structural model is recursive, it too is identified (Rule 6.1). Because the original SR model in Figure

6.6(a) meets both parts of the sufficient two-step rule (Rule 6.9), it is identified, specifically, overidentified.

It is not always possible to determine the identification status of every fully latent SR model using the two-step identification heuristic. For example, suppose that the structural portion of an SR model is nonrecursive such that it does not have all possible disturbance correlations, nor is it block recursive. In this case, the rank condition (Rule 6.3) is not a sufficient condition for identifying the structural model. Therefore, the nonrecursive structural model is "none-of-the-above" concerning identification. Consequently, evaluation of the two-step rule cannot clearly establish whether the original SR model is identified. The same thing can happen when the measurement model of an SR model has both error correlations and complex indicators: If either the measurement or structural portions of an SR model is "none-of-the-above" such that its identification status cannot be clearly established, the two-step rule may be too strict. That is, an SR model of ambiguous identification status may fail the two-step rule but still be identified. Fortunately, many SR models described in the literature have standard measurement models and recursive structural models. In this case, identification status is clear: such SR models are identified.

A HEALTHY PERSPECTIVE ON IDENTIFICATION

Respecification of a structural equation model so that it is identified can at first seem like a shell game: Add this path, drop another, switch an error correlation and—voilà!—the model is identified or—curses!—it is not. Although one obviously needs an identified model, it is crucial to modify models in a judicious manner. That is, any change to the original specification of a model for the sake of identification should be guided by your hypotheses and theory, not by empirical ones. For example, one cannot estimate a model, find that a path coefficient is close to zero, and then eliminate the path in order to identify a model (Kenny et al., 1998). Don't lose sight of the ideas that motivated the analysis in the first place through haphazard specification.

EMPIRICAL UNDERIDENTIFICATION

Although it is *theoretically* possible (that word again) for the computer to derive a set of unique estimates for the parameters of identified models, their analysis can still be foiled by other types of problems. Data-related problems are one such difficulty. For example, extreme collinearity can result in what Kenny (1979) referred to as **empirical underidentification**. For example, if two observed variables are very highly correlated (e.g., r_{XY} = .90), then, practically speaking, they are the same variable. This reduces the effective number of observations below the value of $v\ (v + 1)/2$ (i.e., Rule 5.2). An effective reduction in the number of observations can also shrink the effective value of df_M, perhaps to

less than zero. The good news about this kind of empirical underidentification is that it can be detected through careful data screening.

Other types of empirical underidentification can be more difficult to detect, such as when estimates of certain key paths in a nonrecursive structural model equal a very small or a very high value. Suppose that the coefficient for the path $X_2 \rightarrow Y_2$ in the nonrecursive model of Figure 6.2(b) is about zero. The virtual absence of this path alters the system matrix for the first block of endogenous variables such that the rank of the equation for Y_1 for the model in Figure 6.2(b) without the path $X_2 \rightarrow Y_2$ is zero, which violates the rank condition. You will be asked in an exercise to demonstrate this fact for Figure 6.2(b). Empirical underidentification can affect CFA and SR models, too. Suppose that the estimated factor loading for the path $A \rightarrow X_2$ in the single-factor, three-indicator model of Figure 6.4(b) is close to zero. Practically speaking, this model would resemble the one in Figure 6.4(a) in that factor A has only two indicators, which is too few for a single-factor model. A few additional examples are considered next.

The two-factor model of Figure 6.4(c) may be empirically underidentified if the estimate of the covariance (or correlation) between factors A and B is close to zero. The virtual elimination of the path $A \curvearrowright B$ from this model transforms it into two single-factor, two-indicator models, each of which is underidentified. Measurement models where all indicators load on two factors, such as the classic model for a multitrait-multimethod (MTMM) analysis where each indicator loads on both a trait factor and a method factor (Chapter 9), are especially susceptible to empirical underidentification (Kenny et al., 1998). The identification status of different types of CFA models for MTMM data is considered in Chapter 9. The measurement model in Figure 6.5(f) where indicator X_3 loads on both factors may be empirically underidentified if the absolute estimate of the factor correlation is close to 1.0. Specifically, this extreme collinearity, but now between factors instead of observed variables, can complicate the estimation of X_3's factor loadings. Other possible causes of empirical underidentification include (1) violation of the assumptions of normality or linearity when using normal theory methods (e.g., default ML estimation) and (2) specification errors (Rindskopf, 1984).

MANAGING IDENTIFICATION PROBLEMS

The best advice for avoiding identification problems was given earlier but is worth repeating: Evaluate whether your model is identified right after it is specified but before the data are collected. That is, prevention is better than cure. If you know that your model is in fact identified yet the analysis fails, the source of the problem may be empirical underidentification or a mistake in computer syntax. If a program error message indicates a failure of iterative estimation, another possible diagnosis is poor start values, or initial estimates of model parameters. How to specify better start values is discussed in Chapter 7 for structural models and Chapter 9 for measurement models.

Perhaps the most challenging problem occurs when analyzing a complex model for which no clear identification heuristic exists. This means that whether the model

is actually identified is unknown. If the analysis fails in this case, it may be unclear whether the model is at fault (it is not really identified), the data are to blame (e.g., empirical underidentification), or you made a mistake (syntax error or bad start values). Ruling out a mistake does not resolve the basic ambiguity about identification. Here are some tips on how to cope:

1. A necessary but insufficient condition for the identification of a structural equation model is that an SEM computer can generate a converged solution with no evidence of technical problems such as Heywood cases, or illogical estimates (described in the next chapter). This empirical check can be applied to the actual data. Instead, you can use an SEM computer program as a *diagnostic tool* with made-up data that are anticipated to approximate actual values. This suggestion assumes that the data are not yet collected, which is when the identification question should be addressed. Care must be taken not to generate hypothetical correlations or covariances that are out of bounds (but you can check whether the matrix is positive definite; Chapter 3) or that may result in empirical underidentification. If you are unsure about a particular made-up data matrix, then others with somewhat different but still plausible values can be constructed. The model is then analyzed with the hypothetical data. If a computer program is unable to generate a proper solution, the model may not be identified. Otherwise, it may be identified, but this is not guaranteed. The solution should be subjected to other empirical checks for identification described in Chapter 9, but these checks concern only necessary requirements for identification.
2. A common beginner's mistake in SEM is to specify a complex model of ambiguous identification status and then attempt to analyze it. If the analysis fails (likely), it is not clear what caused the problem. Start instead with a simpler model that is a subset of the whole model and is also one for which the application of heuristics can prove identification. If the analysis fails, the problem is not identification. Otherwise, add parameters to the simpler model one at a time. If the analysis fails after adding a particular effect, try a different order. If these analyses also fail at the same point, then adding the corresponding parameter may cause underidentification. If no combination of adding effects to a basic identified model gets you to the target model, then think about how to respecify the original model in order to identify it and yet still respect your hypotheses.

SUMMARY

It is easy to determine whether recursive path models, standard confirmatory factor analysis models, and structural regression models with recursive structural models and standard measurement models are identified. About all that is needed is to check whether the model degrees of freedom are at least zero, every latent variable has a scale, and every factor has at least two indicators. However, the identification status of nonrecursive structural models or nonstandard measurement models is not always so clear. If

a nonrecursive model does not have all possible disturbance correlations or is not block recursive, there may be no easily applied identification heuristic. There are heuristics for measurement models with either correlated errors or indicators that load on multiple factors, but these rules may not work for more complicated models with both features just mentioned. It is best to avoid analyzing a complex model of ambiguous identification status as your initial model. Instead, first analyze simpler models that you know are identified before adding free parameters. A later chapter (11) deals with identification when means are analyzed in SEM. The next chapter concerns the estimation step.

RECOMMENDED READINGS

The works listed next are all resources for dealing with potential identification problems of more complex models. Rigdon (1995) devised a visual typology for checking whether nonrecursive structural models are identified. See Kenny et al. (1998) for more detail about the identification rules for nonstandard measurement models discussed earlier. Some identification rules by O'Brien (1994) can be applied to measurement models with error correlations where some factors have five or more indicators.

Kenny, D. A., Kashy, D. A., & Bolger, N. (1998). Data analysis in social psychology. In D. Gilbert, S. Fiske, & G. Lindzey (Eds.), *The handbook of social psychology* (Vol. 1, 4th ed., pp. 233–265). Boston, MA: McGraw-Hill.

O'Brien, R. M. (1994). Identification of simple measurement models with multiple latent variables and correlated errors. *Sociological Methodology, 24*, 137–170.

Rigdon, E. E. (1995). A necessary and sufficient identification rule for structural models estimated in practice. *Multivariate Behavioral Research, 30*, 359–383.

EXERCISES

1. Write more specific versions of Rule 5.1 about model parameters for path models, CFA models, and SR models when means are not analyzed.
2. Explain why this statement is generally untrue: The specification $B \rightarrow X_3 = 1.0$ in Figure 6.4(c) assigns to factor B the same scale as that of indicator X_3.
3. Show that the factor models in Figures 6.1(a) and 6.1(b) have the same degrees of freedom.
4. Show for the nonrecursive path model in Figure 6.3 that $df_M = -1$ and also that this model fails both the order condition and the rank condition.
5. Show that the nonrecursive model in Figure 6.3 is identified when the path $X_3 \rightarrow Y_1$ is included in the model.
6. Variable X_3 of Figure 6.5(f) is a complex indicator with loadings on two factors. If the error correlation $E_{X_3} \curvearrowleft E_{X_5}$ is added to this model, would the result-

ing respecified model be identified? If yes, determine whether additional error correlations involving X_3 could be added to the respecified model (i.e., the one with $E_{X_3} \curvearrowleft E_{X_5}$).

7. Suppose that the estimate of the path $X_2 \rightarrow Y_2$ in the block recursive path model of Figure 6.2(b) is close to zero. Show that the virtual absence of this path may result in empirical underidentification of the equation for at least one endogenous variable.

8. Consider the SR model in Figure 6.6(a). If the error correlations $D_B \curvearrowleft D_C$, $E_{X_1} \curvearrowleft E_{Y_1}$, and $E_{X_2} \curvearrowleft E_{Y_2}$ were all added to this model, would the resulting respecified model be identified?

APPENDIX 6.A

Evaluation of the Rank Condition

The starting point for checking the rank condition is to construct a **system matrix**, in which the endogenous variables of the structural model are listed on the left side of the matrix (rows) and all variables in the structural model (excluding disturbances) along the top (columns). In each row, a 0 or 1 appears in the columns that correspond to that row. A 1 indicates that the variable represented by that column has a direct effect on the endogenous variable represented by that row. A 1 also appears in the column that corresponds to the endogenous variable represented by that row. The remaining entries are 0's, and they indicate excluded variables. The system matrix for the model of Figure 6.2(a) with all possible disturbance correlations is presented here (**I**):

$$\begin{array}{c} \\ Y_1 \\ Y_2 \\ Y_3 \end{array} \begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & X_3 & Y_1 & Y_2 & Y_3 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array} \qquad \textbf{(I)}$$

"Reading" this matrix for Y_1 indicates three 1's in its row, one in the column for Y_1 itself, and the others in the columns of variables that, according to the model, directly affect it, X_1 and Y_3. Because X_2, X_3, and Y_2 are excluded from Y_1's equation, the entries in the columns for these variables are all 0's. Entries in the rows for Y_2 and Y_3 are read in a similar way.

The rank condition is evaluated using the system matrix. Like the order condition, the rank condition must be evaluated for the equation of each endogenous variable. The steps to do so for a model with all possible disturbance correlations are outlined next:

1. Begin with the first row of the system matrix (the first endogenous variable). Cross out all entries of that row. Also cross out any column in the system matrix with a 1 in this row. Use the entries that remain to form a new, reduced matrix. Row and column labels are not needed in the reduced matrix.

2. Simplify the reduced matrix further by deleting any row with entries that are all zeros. Also delete any row that is an exact duplicate of another or that can be reproduced by adding other rows together. The number of remaining rows is the rank. (Readers familiar with matrix algebra may recognize this step as the equivalent of elementary row operations to find the rank of a matrix.) For example, consider the following reduced matrix:

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{array}\right] \qquad \textbf{(II)}$$

The third row can be formed by adding the corresponding elements of the first and second rows, so it should be deleted. Therefore, the rank of this matrix (**II**) is 2 instead of 3. *The rank condition is met for the equation of this endogenous variable if the rank of the reduced matrix is greater than or equal to the total number of endogenous variables minus 1.*

3. Repeat steps 1 and 2 for every endogenous variable. *If the rank condition is satisfied for every endogenous variable, then the model is identified.*

Steps 1 and 2 applied to the system matrix for the model of Figure 6.2(a) with all possible disturbance correlations are outlined here (**III**). Note that we are beginning with Y_1:

(III)

	X_1	X_2	X_3	Y_1	Y_2	Y_3
→ Y_1	~~1~~	~~0~~	~~0~~	~~1~~	~~0~~	~~1~~
Y_2	~~0~~	1	0	~~1~~	1	~~0~~
Y_3	~~0~~	0	1	~~0~~	1	~~1~~

→

1	0	1
0	1	1

→ Rank = 2

For step 1, all the entries in the first row of the system matrix (**III**) are crossed out. Also crossed out are three columns of the matrix with a 1 in this row (i.e., those with column headings X_1, Y_1, and Y_3). The resulting reduced matrix has two rows. Neither row has entries that are all zero or can be reproduced by adding other rows together, so the reduced matrix cannot be simplified further. This means that the rank of the equation for Y_1 is 2. This rank exactly equals the required minimum value, which is one less than the total number of endogenous variables in the whole model, or 3 – 1 = 2. The rank condition is satisfied for Y_1.

We repeat this process for the other two endogenous variables for the model of Figure 6.2(a), Y_2 and Y_3. The steps for the remaining endogenous variables are summarized next.

Evaluation for Y_2 (**IV**):

(IV)

	X_1	X_2	X_3	Y_1	Y_2	Y_3
Y_1	1	~~0~~	0	~~1~~	~~0~~	1
→ Y_2	~~0~~	~~1~~	~~0~~	~~1~~	~~1~~	~~0~~
Y_3	0	~~0~~	1	~~0~~	~~1~~	1

→

1	0	1
0	1	1

→ Rank = 2

Evaluation for Y_3 (**V**):

(V)

	X_1	X_2	X_3	Y_1	Y_2	Y_3
Y_1	1	0	~~0~~	1	~~0~~	~~1~~
Y_2	0	1	~~0~~	1	~~1~~	~~0~~
→ Y_3	~~0~~	~~0~~	~~1~~	~~0~~	~~1~~	~~1~~

→

1	0	1
0	1	1

→ Rank = 2

The rank of the equations for each of Y_2 and Y_3 is 2, which exactly equals the minimum required value. Because the rank condition is satisfied for all three endogenous variables of this model, we conclude that it is identified.

The rank condition is evaluated separately for each block of endogenous variables in the block recursive model of Figure 6.2(b). The steps are as follows: First, construct a system matrix for each block. For example, the system matrix for the block that contains Y_1 and Y_2 lists only these variables plus prior variables (X_1 and X_2). Variables of the second block are not included in the matrix for the first block. The system matrix for the second block lists only Y_3 and Y_4 in its rows but represents all of the variables in the whole structural model in its columns. Next, the rank condition is evaluated for the system matrix of each block. These steps are outlined next.

Evaluation for block 1 **(VI)**:

(VI)

		X_1	X_2	Y_1	Y_2			
→	Y_1	~~1~~	~~0~~	~~1~~	~~1~~	→	1	→ Rank = 1
	Y_2	~~0~~	1	~~1~~	~~1~~			

		X_1	X_2	Y_1	Y_2			
	Y_1	1	~~0~~	~~1~~	~~1~~			
→	Y_2	~~0~~	~~1~~	~~1~~	~~1~~	→	1	→ Rank = 1

Evaluation for block 2 **(VII)**:

(VII)

		X_1	X_2	Y_1	Y_2	Y_3	Y_4			
→	Y_3	~~0~~	~~0~~	~~1~~	~~0~~	~~1~~	~~1~~			
	Y_4	0	0	~~0~~	1	~~1~~	~~1~~	→	0 0 1	→ Rank = 1

		X_1	X_2	Y_1	Y_2	Y_3	Y_4			
	Y_3	0	0	1	~~0~~	~~1~~	~~1~~			
→	Y_4	~~0~~	~~0~~	~~0~~	~~1~~	~~1~~	~~1~~	→	0 0 1	→ Rank = 1

Because the rank of the equation of every endogenous variable of each system matrix equals the number of endogenous variables minus 1 (i.e., 2 – 1), the rank condition is met. Thus, the block recursive model of Figure 6.2(b) is identified.